# The Multiplicative Domain in Quantum Error Correction


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## The Big Question

If we want to send some quantum data through a given noisy channel, how can we do it so that the information is preserved?

## Mathematical Basics

Let $\mathcal{H}$ be a finite-dimensional Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the set of linear operators on $\mathcal{H}$.

- A completely positive (CP) trace-preserving linear map $\mathcal{E} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ is called a *quantum channel*.
- $\mathcal{E}$ is said to be *unital* if $\mathcal{E}(I_{\mathcal{H}}) = I_{\mathcal{H}}$.
- $\mathcal{A}$ and $\mathcal{B}$ are called *subsystems* of $\mathcal{H}$ if we can write $\mathcal{H} = (\mathcal{A} \otimes \mathcal{B}) \oplus (\mathcal{A} \otimes \mathcal{B})^{\perp}$.

## Correctable Subsystems

Given a quantum channel $\mathcal{E}$, a subsystem $\mathcal{B}$ of $\mathcal{H}$ is said to be a *correctable subsystem* [1] if there exists a quantum channel $\mathcal{R}$ such that

$$\forall \sigma^A, \sigma^B \; \exists \tau^A \quad \text{s.t.} \quad \mathcal{R} \circ \mathcal{E}(\sigma^A \otimes \sigma^B) = \tau^A \otimes \sigma^B.$$

- The channel $\mathcal{R}$ is known as the *recovery operation*.
- We can decompose $\mathcal{R}$ into a two step form:
  1. Perform a projective measurement.
  2. Conjugate by a unitary (which can depend on the result of the measurement).
- If $\mathcal{R} = id_{\mathcal{H}}$ is the identity map then $\mathcal{B}$ is called a *noiseless subsystem* [2].

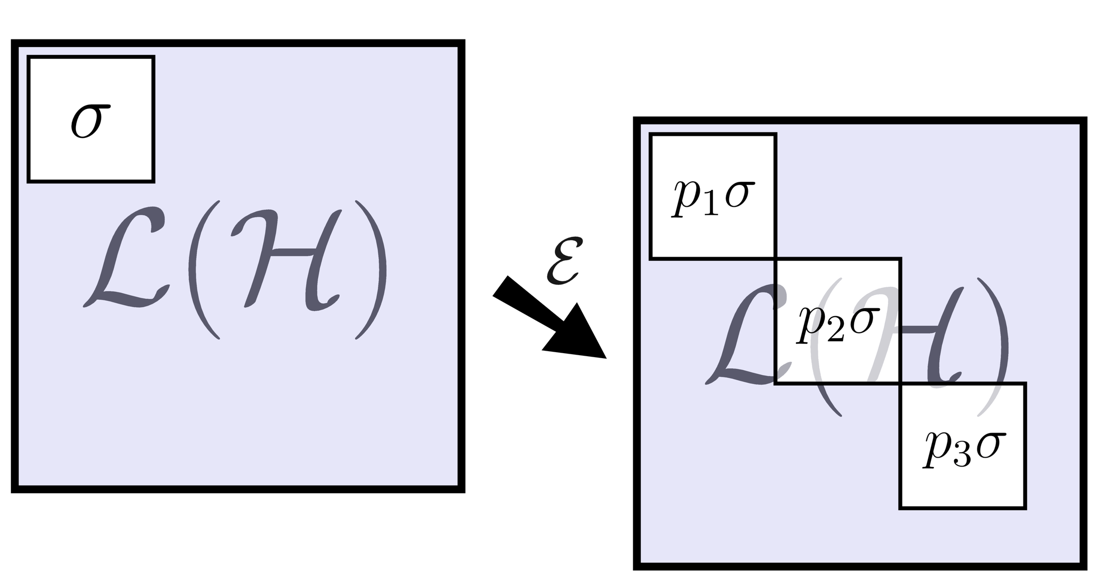


Figure 1: A correctable subsystem, depicted as a sub-block of the operators acting on the Hilbert space. To correct the error, project onto one of the three resulting sub-blocks and then conjugate by a unitary.

## Unitarily-Correctable Codes

A correctable subsystem $\mathcal{B}$ is said to be a *unitarily-correctable code* (UCC) for $\mathcal{E}$ if the recovery operation is simply a conjugation-by-unitary channel $\mathcal{U}(\cdot) := U(\cdot)U^*$.

- Since finding correctable subsystems in full generality is an extremely difficult problem, restricting our attention to unitarily-correctable codes seems potentially wise.
- These codes are of physical interest; they are codes in which the two-step process of recovery only involves the conjugation-by-unitary step (and not the projective measurement step).
- It has been shown [3] that if a quantum channel $\mathcal{E}$ is unital, then we can unambiguously define *the unitarily-correctable code algebra* of $\mathcal{E}$, denoted $UCC(\mathcal{E})$, to be the albebra composed of the direct sum of all of the unitarily-correctable codes.
- In terms of Figure 1, unitarily-correctable codes are those for which $p_1 = 1$ and $p_2 = p_3 = 0$ (i.e., there is just one block on the right).

## The Multiplicative Domain

The *multiplicative domain* of $\mathcal{E}$ [4], denoted $MD(\mathcal{E})$, is defined to be the following set:

$$\{a \in \mathcal{L}(\mathcal{H}) : \mathcal{E}(a)\mathcal{E}(b) = \mathcal{E}(ab) \text{ and } \mathcal{E}(b)\mathcal{E}(a) = \mathcal{E}(ba) \; \forall \, b \in \mathcal{L}(\mathcal{H})\}.$$

- $\mathcal{E}$ behaves particularly nicely when restricted to $MD(\mathcal{E})$ (as a $*$-homomorphishm, in fact).
- $MD(\mathcal{E})$ was first studied by operator theorists over thirty years ago.
- $MD(\mathcal{E})$ is an algebra, and hence [5] is unitarily equivalent to a direct sum of tensor blocks:

$$MD(\mathcal{E}) \cong \oplus_k (I_{\mathcal{A}_k} \otimes \mathcal{L}(\mathcal{B}_k)) \oplus 0_{\mathcal{K}}. \qquad (1)$$

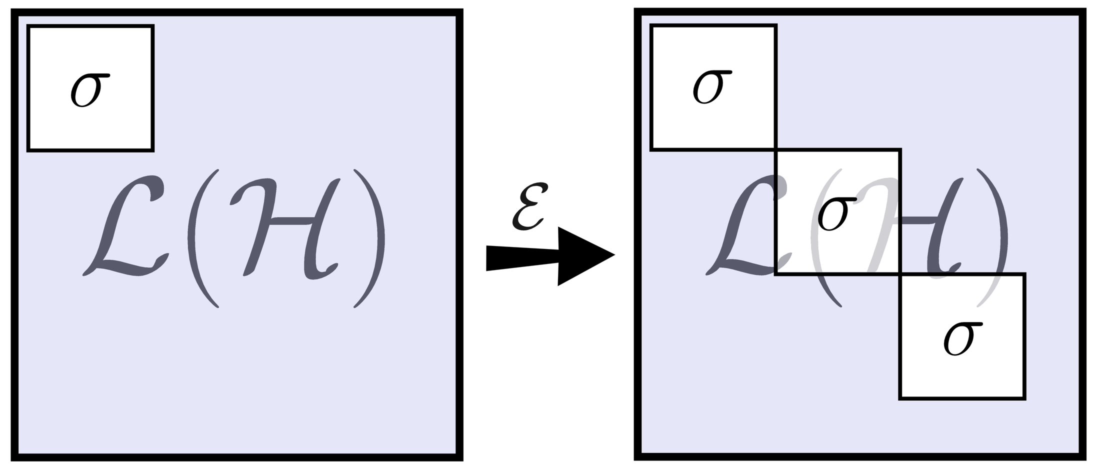


Figure 2: The action of a quantum channel on its multiplicative domain.

## The Great Connection

By looking at Figures 1 and 2, we expect that there might be some connection between correctable subsystems for a channel $\mathcal{E}$ and its multiplicative domain. Indeed, one of our main results is that the two situations coincide when $\mathcal{E}$ is unital and the subsystem is unitarily-correctable.

## Main Result

**Theorem.** *Let $\mathcal{E}$ be a unital quantum channel. Then $MD(\mathcal{E}) = UCC(\mathcal{E})$.*

- This theorem says that when we write $MD(\mathcal{E})$ in the form of Equation (1), the $\mathcal{B}_k$'s are exactly the unitarily-correctable codes for $\mathcal{E}$.
- When $\mathcal{E}$ is not unital, $MD(\mathcal{E})$ in general only captures a subclass of the unitarily-correctable codes for $\mathcal{E}$.
- Because $MD(\mathcal{E})$ is easy to compute, this provides a concrete method of finding some UCCs.

## Generalization

In the same spirit as the multiplicative domain, we can define "generalized multiplicative domains" for channels by requiring not that the channel be multiplicative with itself, but rather that it be multiplicative with some $*$-homomorphism.

- Generalized multiplicative domains capture *all* correctable codes for *arbitrary* channels.
- Unlike the multiplicative domain, these algebras in general are very difficult to compute.

## Conclusions and Outlook

This characterization provides a simple way to find all unitarily-correctable codes for unital channels and even some codes for non-unital channels. General correctable subsystems can be characterized in terms of algebras that are analogous to the multiplicative domain, though in general it is not clear how to calculate them – further research in this area would be of great interest.

## For Further Information

For the details of our work:

- Choi, M.-D., Johnston, N., and Kribs, D. W.. Journal of Physics A: Mathematical and Theoretical **42**, 245303 (2009).
- Johnston, N., and Kribs, D. W., *Generalized Multiplicative Domains and Quantum Error Correction* (2009, preprint).

Preprints and this poster can be downloaded from:

- www.arxiv.org
- www.nathanieljohnston.com

## References


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[2] P. Zanardi, M. Rasetti, Phys. Rev. Lett. **79**, 3306 (1997).

[3] D. W. Kribs, R. W. Spekkens, Phys. Rev. A **74**, 042329 (2006).

[4] M.-D. Choi, Illinois J. Math., **18** (1974), 565-574.

[5] K. R. Davidson, *$C^*$-algebras by example*, Fields Institute Monographs, 6. American Mathematical Society, Providence, RI, 1996.


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