Antidistinguishability and k-Incoherence

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Distinguishability

A **pure state** is a unit vector $|\phi\rangle \in \mathbb{C}^d$.

If we are given a single copy of an arbitrary pure state (in a lab, not on paper), we cannot figure out exactly which one was given to us: measuring it gives some information but causes the state to collapse.

However, if we are given extra information about the state, sometimes we *can* figure out which state was given to us...

Distinguishability

Suppose we are given (on paper) a set of potential states:

$$S = \big\{|\phi_0\rangle, |\phi_1\rangle, \ldots, |\phi_{n-1}\rangle\big\} \subset \mathbb{C}^d.$$

Then we are given (in a lab) one of those n states.

Theorem

*It is possible to determine which $|\phi_j\rangle$ was given to us (i.e., S is **distinguishable**) if and only if the members of S are mutually orthogonal (i.e., $\langle\phi_i|\phi_j\rangle = 0$ whenever $i \neq j$).*

Antidistinguishability

What if, instead of wanting to determine which state from S was given to us, we just want to determine some state from S that was *not* given to us?

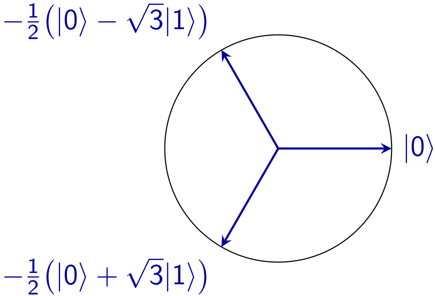
In other words, we want to determine whether or not S is **antidistinguishable**.

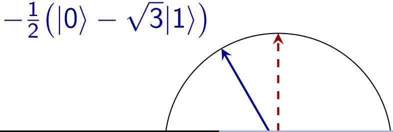
- If S is distinguishable then it is antidistinguishable.
- If $n = 2$ then S is distinguishable iff S is antidistinguishable.
- If $n \geq 3$ then there are antidistinguishable sets that are not distinguishable...

Antidistinguishability Example

For example, consider the set of "trine" states:

$$S = \left\{ |0\rangle, \quad -\tfrac{1}{2}\big(|0\rangle + \sqrt{3}|1\rangle\big), \quad -\tfrac{1}{2}\big(|0\rangle - \sqrt{3}|1\rangle\big) \right\} \subset \mathbb{C}^2.$$





Some Inner Product Bounds

Fact: Whether or not a set S is antidistinguishable depends only on the inner products between the $|\phi_j\rangle$'s.

If the inner products are large then S is not antidistinguishable:

Theorem (Bandyopadhyay–Jain–Oppenheim–Perry '14)

Let $n \geq 2$ be an integer and let $S = \{|\phi_0\rangle, |\phi_1\rangle, \ldots, |\phi_{n-1}\rangle\}$. If

$$\big|\langle\phi_i|\phi_j\rangle\big| > \frac{n-2}{n-1} \quad \textit{for all} \quad 0 \leq i \neq j \leq n-1$$

then S is not antidistinguishable. Furthermore, this bound is tight.

A Conjecture

Conversely, if the inner products are small (e.g., all less than 1/2) then S is antidistinguishable.

Conjecture (Havlívcek–Barrett '20)

Let $n \geq 2$ be an integer and let $S = \{|\phi_0\rangle, |\phi_1\rangle, \ldots, |\phi_{n-1}\rangle\}$. If

$$\big|\langle\phi_i|\phi_j\rangle\big| \leq \frac{n-2}{n-1} \quad \text{for all} \quad 0 \leq i \neq j \leq n-1$$

then S is antidistinguishable.

This conjecture is true (and tight) for $n \leq 3$.

Refutation of the Conjecture

Via a couple years of computer search, the $n = 4$ case of this conjecture was recently disproved by Russo and Sikora:

- The conjecture says that if $|\langle\phi_i|\phi_j\rangle| \leq 2/3 = 0.6666\ldots$ for all $i \neq j$ then S is antidistinguishable.
- They numerically found a non-antidistinguishable set of states with $|\langle\phi_i|\phi_j\rangle| \leq 0.6451$ for all $i \neq j$.

So what is the "correct" bound for $n = 4$?

k-Incoherence

We're going to (seemingly) switch gears for a moment.

- A matrix $X \in M_n(\mathbb{C})$ is **positive semidefinite (PSD)** if it is Hermitian ($X^* = X$) and has non-negative eigenvalues.

- A matrix $X \in M_n(\mathbb{C})$ is called **k-incoherent** if we can write

$$X = \sum_j X_j$$

for some PSD matrices that are 0 outside of a single $k \times k$ principal submatrix.

k-Incoherence

For example...

- If $k = 1$, a matrix is 1-incoherent if and only if it is PSD and diagonal.
- If $k = n$, a matrix is n-incoherent if and only if it is PSD.
- If $n = 3$, $k = 2$ then the following matrix is 2-incoherent:

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & -1 \\ 2 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Connection with $(n-1)$-Incoherence

It turns out that antidistinguishability is equivalent to k-incoherence in the $k = n-1$ case:

Theorem (J.–Russo–Sikora '23)

Let $n \geq 2$ be an integer and let $S = \{|\phi_0\rangle, |\phi_1\rangle, \ldots, |\phi_{n-1}\rangle\}$. Then S is antidistinguishable if and only if the Gram matrix

$$G = \begin{bmatrix} 1 & \langle\psi_0|\psi_1\rangle & \cdots & \langle\psi_0|\psi_{n-1}\rangle \\ \langle\psi_1|\psi_0\rangle & 1 & \cdots & \langle\psi_1|\psi_{n-1}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\psi_{n-1}|\psi_0\rangle & \langle\psi_{n-1}|\psi_1\rangle & \cdots & 1 \end{bmatrix}$$

is $(n-1)$-incoherent.

Connection with $(n-1)$-Incoherence

Cool! We can now apply all sorts of things that are already known (or mostly known) about $(n-1)$-incoherence. Example:

Theorem (J.–Moein–Pereira–Plosker–Russo–Sikora '23)

Let $n \geq 2$ be an integer, let $S = \{|\phi_0\rangle, |\phi_1\rangle, \ldots, |\phi_{n-1}\rangle\}$, and let $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1}$ be the eigenvalues of the Gram matrix G. If

$$\sqrt{\lambda_0} \leq \sum_{j=1}^{n-1} \sqrt{\lambda_j}$$

then G is $(n-1)$-incoherent, so S is antidistinguishable. Furthermore, this inequality is tight.

Connection with $(n-1)$-Incoherence

The previous theorem has the following even simpler-to-use corollary:

Corollary (J.–Russo–Sikora '23)

Let $n \geq 2$ be an integer, let $S = \{|\phi_0\rangle, |\phi_1\rangle, \ldots, |\phi_{n-1}\rangle\}$, and let G be the Gram matrix. If

$$\|G\|_{\mathrm{F}} \leq \frac{n}{\sqrt{2}}$$

then G is $(n-1)$-incoherent, so S is antidistinguishable. Furthermore, this inequality is tight.

Correction of the Conjecture

The previous corollary immediately give us the following correction of the antidistinguishability conjecture:

Theorem (J.–Russo–Sikora '23)

Let $n \geq 2$ be an integer and let $S = \{|\phi_0\rangle, |\phi_1\rangle, \ldots, |\phi_{n-1}\rangle\}$. If

$$\big|\langle\phi_i|\phi_j\rangle\big| \leq \frac{1}{\sqrt{2}}\sqrt{\frac{n-2}{n-1}} \quad \text{for all} \quad 0 \leq i \neq j \leq n-1$$

then S is antidistinguishable.

This bound is tight for $n = 2$, $n = 3$, and $n = 4$.

Unknown if it's tight for $n \geq 5$.

Thank You!

Thank you!

k-incoherence: Physical Review A, 106:052417, 2022
arXiv:2205.05110

antidistinguishability: coming soon