

# Approximating Quantum States by States of Low Rank

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# Pure States

A **pure quantum state** is a unit vector  $|\phi\rangle \in \mathbb{C}^d$ .

Actually, no, a pure quantum state is only defined up to global phase, so it's more accurate to call  $|\phi\rangle\langle\phi|$  (i.e., the orthogonal projection onto  $\text{span}(|\phi\rangle)$ ) a pure state.

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# Mixed States

Mixed quantum states arise from just removing the rank restriction.

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# Mixed States (Boo!)

Mixed states are often not actually desired, and are usually (always?) much harder to work with than pure states. For example:

- Determining whether a state is entangled? Easy for pure, NP-hard for mixed.
- Determining the resourcefulness (in most resource theories, including entanglement,  $k$ -coherence, and magic) of a pure state is easy, but doing so for a mixed state is hard.
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# Approximating Mixed States by Pure States

Don't like the mixed state  $\rho$  that you have? Try this:

Step 1: Find the closest pure state  $|\phi\rangle\langle\phi|$  to  $\rho$ .

Step 0: Decide what "closest" means (i.e., pick a metric to measure distance between matrices).

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Depending on what you're trying to do, this might work fairly well. And there is a nice (old!) theorem that tells us how to find  $|\phi\rangle\langle\phi|$  from  $\rho$ :

Theorem (Consequence of Mirsky's inequality (1960))

*Let  $\rho \in \mathcal{M}_d$  be a mixed state and let  $\|\cdot\| : \mathcal{M}_d \rightarrow \mathbb{R}$  be a unitary similarity invariant norm (i.e.,  $\|UXU^*\| = \|X\|$  for all  $X \in \mathcal{M}_d$ ).*

*Then the quantity*

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How does our earlier methodology change in this more general setting?

**Step 0:** Fix an acceptable rank  $r$  and decide what “closest” means (i.e., pick a metric to measure distance between matrices).

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If we didn't have the requirement that  $\text{Tr}(\sigma) = 1$ , this problem would have a well-known solution:

Theorem (Basically the Eckart–Mirsky–Young theorem)

*Let  $\rho \in \mathcal{M}_d$  be a mixed state and let  $\|\cdot\| : \mathcal{M}_d \rightarrow \mathbb{R}$  be a unitary similarity invariant norm.*

*Then the quantity*

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- **Trace norm:**  $\|X\|_{\text{tr}} = \sum_{j=1}^d \sigma_j(X).$

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In particular, they showed the following:

Theorem (Ezzell–Holmes–Coles, 2022)

Let  $r \in \{1, \dots, d-1\}$  be a positive integer,  $\|\cdot\| : \mathcal{M}_d \rightarrow \mathbb{R}$  be the trace norm or Frobenius norm, and  $\rho \in \mathcal{M}_d$  be a mixed state with spectral decomposition  $\rho = \sum_{j=1}^d x_j v_j v_j^*$  ( $x_1 \geq \dots \geq x_d \geq 0$ ).

Define

$$\gamma = \frac{1}{r} \sum_{j=r+1}^d x_j \quad \text{and} \quad \sigma = \sum_{j=1}^r (x_j + \gamma) v_j v_j^*.$$

Then  $\|\rho - \sigma\| \leq \|\rho - \tau\|$  for all mixed states  $\tau$  with rank at most  $r$ .

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## Other Unitary Similarity Invariant Norms

Our first main result shows that the same is true for all unitary similarity invariant norms:

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## The Farthest-From-Low-Rank State: Maximally Mixed?

**Follow-up question:** Which mixed state is farthest from the set of low-rank states?

In other words, which mixed state maximizes the quantity

$$D_r(\rho) := \min_{\sigma} \{ \|\rho - \sigma\| \},$$

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# The Maximally Mixed State

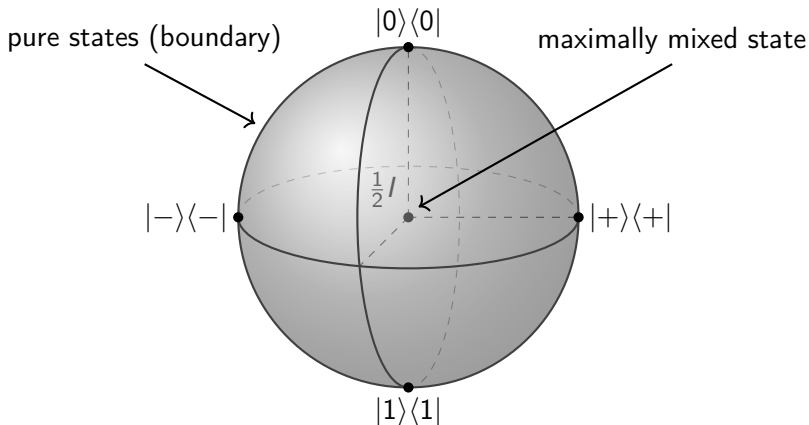


Figure: The Bloch ball (i.e., mixed states in  $\mathcal{M}_2$ ).

# The Maximally Mixed State

This guess is correct in the simple cases:

Proposition (J.-Li, 2026)

*Let  $r \in \{1, \dots, d-1\}$  be a positive integer and let  $\|\cdot\| : \mathcal{M}_d \rightarrow \mathbb{R}$  be a unitary similarity invariant norm.*

*Then  $D_r(\rho)$  is maximized by the maximally mixed state  $\rho = \frac{1}{d}I$  in the following cases:*

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A bit surprisingly, this guess is *not* correct in general.

For  $p \in [1, \infty)$ , the Schatten  $p$ -norm is  $\|X\|_p = \sqrt[p]{\sum_{j=1}^d \sigma_j(X)^p}$ .

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Suppose  $p \in [1, \infty)$  and  $\|\cdot\|$  is the Schatten  $p$ -norm. The following are equivalent:

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For example, if  $p = 3/2$  and  $r = 1$  then  $D_r(\rho)$  (measured in the Schatten  $3/2$ -norm) is maximized by the maximally mixed state  $\rho = \frac{1}{d}I$  when  $d \leq 14$ , but not when  $d \geq 15$ .

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Then  $D_r(\rho)$  has a maximizer of the form

$$\rho = \frac{1}{m} I_m \oplus O_{d-m}$$

for some  $m \in \{r+1, \dots, d\}$ .

That is, the worst-case state is always maximally mixed on some subspace of dimension  $m > r$ , but not necessarily on the whole space.

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### Theorem (J.-Li, 2026)

Let  $r \in \{1, \dots, d-1\}$  be a positive integer and let  $\|\cdot\| : \mathcal{M}_d \rightarrow \mathbb{R}$  be a unitary similarity invariant norm.

Then  $D_r(\rho)$  has a maximizer of the form

$$\rho = \frac{1}{m} I_m \oplus O_{d-m}$$

for some  $m \in \{r+1, \dots, d\}$ .

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